18.1 Two Variables, One Equality Constraint

Consider the problem of maximizing (or minimizing) a function $f(x, y)$ when x and y are restricted to satisfy an equation $g(x, y) = c$. In case we want to maximize $f(x, y)$, the problem is

$$\max_{x,y} f(x, y) \text{ subject to } g(x, y) = c \qquad [18.2]$$

Problem [18.2] can be given a geometric interpretation, as in Fig. 18.1.

[1]The method is named after its discoverer, the French mathematician Joseph Louis Lagrange (1736–1813). The Danish economist Harald Westergaard seems to have been the first who used it in economics, in 1876. (See Thorkild Davidsen, "Westergaard, Edgeworth and the use of Lagrange multipliers in economics," *Economic Journal* 96 (1986): 808–811.)

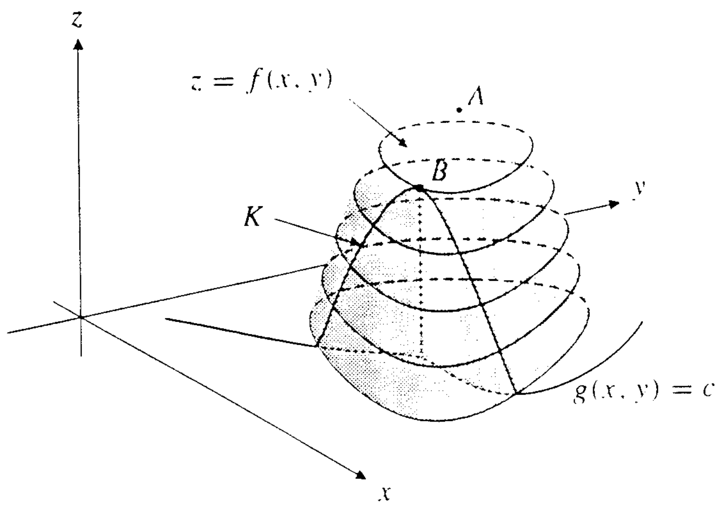

FIGURE 18.1

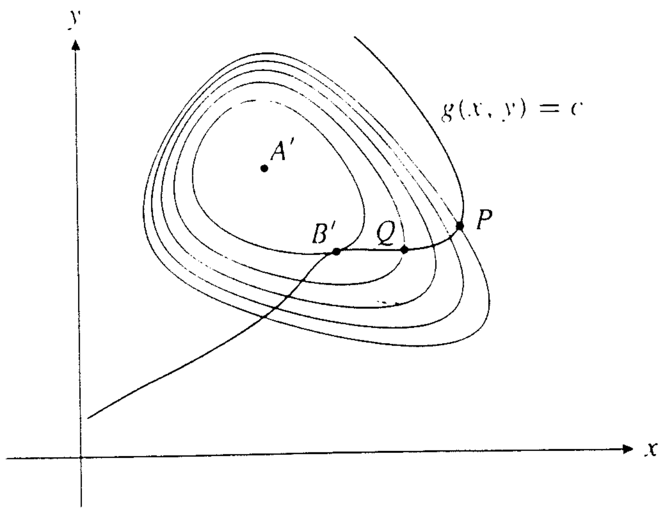

FIGURE 18.2

The graph of f is a surface like a bowl, whereas the equation $g(x, y) = c$ represents a curve in the xy-plane. The curve K on the bowl is the one that lies directly above the curve $g(x, y) = c$. (The latter curve is the projection of K onto the xy-plane.) Maximizing $f(x, y)$ without taking the constraint into account gets us to the peak A in Fig. 18.1. The solution to problem [18.2], however, is at B, which is the highest point on the curve K. If we think of the graph of f as representing a mountain, and K as a mountain path, then we seek the highest point on the path, which is B. Analytically, the problem is to find the coordinates of B.

In Fig. 18.2, we show some of the level curves for f, and also indicate the constraint curve $g(x, y) = c$. Now A' represents the point at which $f(x, y)$ has its unconstrained (free) maximum. The closer a level curve of f is to point A', the higher is the value of f along that level curve. We are seeking that point on the constraint curve $g(x, y) = c$ where f has its highest value. If we start at point P on the constraint curve and move along that curve toward A', we encounter level curves with higher and higher values of f. Obviously, point Q indicated in Fig. 18.2 is not the point on $g(x, y) = c$ at which f has its highest value, because the constraint curve passes *through* the level curve of f at that point. Therefore, we can proceed along the constraint curve and attain higher values of f. However, when we reach point B', we cannot go any higher. It is intuitively clear that B' is the point with the property that the constraint curve touches (without intersecting) a level curve for f. This observation implies that the slope of the tangent to the curve $g(x, y) = c$ at (x, y) is equal to the slope of the tangent to the level curve of f at that point.

Recall from Section 16.3 that the slope of the level curve $F(x, y) = c$ is given by $dy/dx = -F_1'(x, y)/F_2'(x, y)$. Thus, the condition that the slope of the tangent to $g(x, y) = c$ is equal to the slope of a level curve for $f(x, y)$ can be expressed analytically as follows:[2]

$$-g_1'(x, y)/g_2'(x, y) = -f_1'(x, y)/f_2'(x, y)$$

[2]Disregard for a moment points (x, y) at which one or both partials of f and g with respect to y vanish. See Theorem 18.1 for a precise result.

or

$$\frac{f_1'(x, y)}{f_2'(x, y)} = \frac{g_1'(x, y)}{g_2'(x, y)} \qquad [18.3]$$

A corresponding argument for the problem of minimizing $f(x, y)$ subject to $g(x, y) = c$ gives the same condition [18.3]. It follows that a necessary condition for (x, y) to solve problem [18.2] (or the corresponding minimization problem) is that (x, y) satisfies both [18.3] and $g(x, y) = c$. These give two equations for determining the two unknowns x and y.

18.2 The Lagrange Multiplier Method

Recall the constrained optimization problem in [18.2], which is to maximize $f(x, y)$ subject to $g(x, y) = c$. The first-order condition [18.3] can be expressed in a way that is easy both to remember and generalize. First, rearrange [18.3] to obtain

$$\frac{f_1'(x, y)}{g_1'(x, y)} = \frac{f_2'(x, y)}{g_2'(x, y)} \qquad [*]$$

If (x_0, y_0) solves problem [18.2], then the left- and right-hand sides of [*] are equal at (x_0, y_0). The common value λ of these fractions is called a **Lagrange multiplier**, and equation [*] can then be expressed as

$$f_1'(x, y) - \lambda g_1'(x, y) = 0, \qquad f_2'(x, y) - \lambda g_2'(x, y) = 0 \qquad [18.4]$$

Now define the **Lagrangean function** $\mathcal{L}$ by

$$\mathcal{L}(x, y) = f(x, y) - \lambda(g(x, y) - c) \qquad [18.5]$$

The partials of $\mathcal{L}(x, y)$ with respect to x and y are $\mathcal{L}_1'(x, y) = f_1'(x, y) - \lambda g_1'(x, y)$ and $\mathcal{L}_2'(x, y) = f_2'(x, y) - \lambda g_2'(x, y)$, respectively. Thus, Equation [18.4] is the

first-order condition expressing the requirement that the partials of L vanish. This argument supports the following procedure:

> **The Lagrangean Method**
>
> To find the solutions of the problem
>
> $$\max(\min)\ f(x, y) \text{ subject to } g(x, y) = c$$
>
> proceed as follows:
>
> **1.** Write down the Lagrangean function
>
> $$\mathrm{L}(x, y) = f(x, y) - \lambda\big(g(x, y) - c\big)$$
>
> where λ is a constant.
>
> **2.** Differentiate L with respect to x and y, and equate the partials to 0.
>
> **3.** The two equations in **2** together with the constraint yield the following three equations:
>
> $$f_1'(x, y) = \lambda g_1'(x, y)$$
> $$f_2'(x, y) = \lambda g_2'(x, y)$$
> $$g(x, y) = c$$
>
> **4.** Solve these three equations for the three unknowns x, y, and λ.

This method will in general give us all pairs of numbers (x, y) that can possibly solve the problem. As a bonus, we get the corresponding value of the Lagrange multiplier λ. We shall see shortly that λ has a very interesting interpretation that is useful in many economic optimization problems.[3]

Example 18.3

Use Lagrange's method for the problem in Example 18.1.

Solution The Lagrangean is

$$\mathrm{L}(x, y) = xy - \lambda(2x + y - m)$$

So the first-order conditions for the solution of the problem are

$$\mathrm{L}_1'(x, y) = y - 2\lambda = 0, \qquad \mathrm{L}_2'(x, y) = x - \lambda = 0, \qquad 2x + y = m \qquad [*]$$

[3]Some prefer to consider the Lagrangean as a function of three variables, $\mathrm{L}(x, y, \lambda)$. Then $\partial \mathrm{L}/\partial\lambda = -[g(x, y) - c]$, so equating this partial to 0 yields the constraint $g(x, y) = c$. Later in Section 18.8, when inequality constraints are being discussed, some dangers of this procedure will be pointed out.

The first two equations imply that $y = 2\lambda$ and $x = \lambda$. So $y = 2x$. Inserting this into the constraint yields $2x + 2x = m$. Therefore, $x = m/4$, $y = m/2$, and $\lambda = x = m/4$. This is the same solution for x and y as we found in Example 18.1.